

NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR FUNCTIONS WHOSE DERIVATIVES ABSOLUTE VALUES ARE QUASI-CONVEX

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ABSTRACT. In this paper we establish some estimates of the right hand side of a Hermite-Hadamard type inequality in which some quasi-convex functions are involved.

1. INTRODUCTION

Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following inequality, known as the *Hermite-Hadamard* inequality for convex functions, holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

In recent years many authors have established several inequalities connected to *Hermite-Hadamard's* inequality. For recent results, refinements, counterparts, generalizations and new *Hermite-Hadamard* type inequalities see [2], [4] and [5].

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a, b] \rightarrow \mathbb{R}$ is said to be quasi-convex on $[a, b]$ if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\},$$

for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [3]).

Recently, D.A. Ion [3] established two inequalities for functions whose first derivatives in absolute value are quasi-convex. Namely, he obtained the following results:

Theorem 1. *Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{b-a}{4} \{ \max\{|f'(a)|, |f'(b)|\} \}.$$

Theorem 2. *Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|^{\frac{p}{p-1}}$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} \left(\max\left\{ |f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}}.$$

2000 *Mathematics Subject Classification.* Mathematics Subject Classification. 26A51, 26D10.

Key words and phrases. quasi-convex functions, hölder inequality, power mean inequality.

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In [1], Alomari et al. obtained the following results.

Theorem 3. *Let $f : I^\circ \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$(1.1) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{b-a}{8} \left[\max \left\{ \left| f' \left(\frac{a+b}{2} \right) \right|, |f'(a)| \right\} + \max \left\{ \left| f' \left(\frac{a+b}{2} \right) \right|, |f'(b)| \right\} \right].$$

Theorem 4. *Let $f : I^\circ \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^{\frac{p}{p-1}}$ is quasi-convex on $[a, b]$, for $p > 1$ then the following inequality holds:*

$$(1.2) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{b-a}{4} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\left(\max \left\{ \left| f' \left(\frac{a+b}{2} \right) \right|^{\frac{p}{p-1}}, |f'(a)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}} + \left(\max \left\{ \left| f' \left(\frac{a+b}{2} \right) \right|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}} \right].$$

Theorem 5. *Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is quasi-convex on $[a, b]$, $q \geq 1$, then the following inequality holds:*

$$(1.3) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{b-a}{8} \left[\left(\max \left\{ \left| f' \left(\frac{a+b}{2} \right) \right|^q, |f'(a)|^q \right\} \right)^{\frac{1}{q}} + \left(\max \left\{ \left| f' \left(\frac{a+b}{2} \right) \right|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \right].$$

The main purpose of this study is to generalize the Theorem 3, Theorem 4 and Theorem 5 for quasi-convex functions using the new Lemma.

2. HERMITE-HADAMARD TYPE INEQUALITIES

Lemma 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\begin{aligned} \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du &= \frac{(x-a)^2}{b-a} \int_0^1 (t-1) f'(tx + (1-t)a) dt \\ &\quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) f'(tx + (1-t)b) dt. \end{aligned}$$

Theorem 6. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds:

$$\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{(x-a)^2}{2(b-a)} \max\{|f'(x)|, |f'(a)|\} \\ + \frac{(b-x)^2}{2(b-a)} \max\{|f'(x)|, |f'(b)|\}.$$

Proof. From Lemma 1, we have

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| dt \\ & \quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| dt \\ & \leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) \max\{|f'(x)|, |f'(a)|\} dt \\ & \quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) \max\{|f'(x)|, |f'(b)|\} dt \\ & = \frac{(x-a)^2}{b-a} \max\{|f'(x)|, |f'(a)|\} \int_0^1 (1-t) dt \\ & \quad + \frac{(b-x)^2}{b-a} \max\{|f'(x)|, |f'(b)|\} \int_0^1 (1-t) dt \\ & = \frac{(x-a)^2}{2(b-a)} \max\{|f'(x)|, |f'(a)|\} \\ & \quad + \frac{(b-x)^2}{2(b-a)} \max\{|f'(x)|, |f'(b)|\}, \end{aligned}$$

which completes the proof. \square

Remark 1. In Theorem 6, if we choose $x = \frac{a+b}{2}$, we obtain (1.1) inequality.

Theorem 7. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|^{\frac{p}{p-1}}$ is quasi-convex on $[a, b]$, $p > 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\ & \leq \frac{(x-a)^2}{b-a} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\max\left\{ |f'(x)|^{\frac{p}{p-1}}, |f'(a)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}} \\ & \quad + \frac{(b-x)^2}{b-a} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\max\left\{ |f'(x)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}} \end{aligned}$$

where $q = p/(p-1)$.

Proof. From Lemma 1 and using well known Hölder inequality, we have

$$\begin{aligned}
& \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\
& \leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| dt \\
& \quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| dt \\
& \leq \frac{(x-a)^2}{b-a} \left(\int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(tx + (1-t)a)|^{\frac{p}{p-1}} dt \right)^{\frac{p-1}{p}} \\
& \quad + \frac{(b-x)^2}{b-a} \left(\int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(tx + (1-t)b)|^{\frac{p}{p-1}} dt \right)^{\frac{p-1}{p}} \\
& \leq \frac{(x-a)^2}{b-a} \left(\int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \max \left\{ |f'(x)|^{\frac{p}{p-1}}, |f'(a)|^{\frac{p}{p-1}} \right\} dt \right)^{\frac{p-1}{p}} \\
& \quad + \frac{(b-x)^2}{b-a} \left(\int_0^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \max \left\{ |f'(x)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} dt \right)^{\frac{p-1}{p}} \\
& = \frac{(x-a)^2}{b-a} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\max \left\{ |f'(x)|^{\frac{p}{p-1}}, |f'(a)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}} \\
& \quad + \frac{(b-x)^2}{b-a} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\max \left\{ |f'(x)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}}
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, which completes the proof. \square

Remark 2. In Theorem 7, if we choose $x = \frac{a+b}{2}$, we obtain (1.2) inequality.

Theorem 8. Let $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is quasi-convex on $[a, b]$, $q \geq 1$, then the following inequality holds:

$$\begin{aligned}
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| & \leq \frac{(x-a)^2}{2(b-a)} \left(\max \left\{ |f'(x)|^q, |f'(a)|^q \right\} \right)^{\frac{1}{q}} \\
& \quad + \frac{(b-x)^2}{2(b-a)} \left(\max \left\{ |f'(x)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 1 and using the well known power mean inequality, we have

$$\begin{aligned}
& \left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\
& \leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| dt \\
& \quad + \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| dt \\
& \leq \frac{(x-a)^2}{b-a} \left(\int_0^1 (1-t) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t) |f'(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\
& \quad + \frac{(b-x)^2}{b-a} \left(\int_0^1 (1-t) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t) |f'(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is quasi-convex we have

$$\int_0^1 (1-t) |f'(tx + (1-t)a)|^q dt \leq \frac{1}{2} \max\{|f'(x)|^q, |f'(a)|^q\}$$

and

$$\int_0^1 (1-t) |f'(tx + (1-t)b)|^q dt \leq \frac{1}{2} \max\{|f'(x)|^q, |f'(b)|^q\}.$$

Therefore, we have

$$\begin{aligned}
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| & \leq \frac{(x-a)^2}{2(b-a)} (\max\{|f'(x)|^q, |f'(a)|^q\})^{\frac{1}{q}} \\
& \quad + \frac{(b-x)^2}{2(b-a)} (\max\{|f'(x)|^q, |f'(b)|^q\})^{\frac{1}{q}}.
\end{aligned}$$

□

Remark 3. In Theorem 8, if we choose $x = \frac{a+b}{2}$, we obtain (1.3) inequality.

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